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SOLVING AN INTEGRAL EQUATION VIA INTUITIONISTIC FUZZY BIPOLAR METRIC SPACES

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Abstract *In this paper, we introduce the notion of intuitionistic fuzzy bipolar metric space and prove fixed point theorems. Our results are extension or generalisation of results proved in the literature. The derived results are substantiated with suitable example and an application.*

Key words: *Intuitionistic Fuzzy Bipolar Metric Space; Fixed Point Results; c.*

1. Introduction

Zadeh (Zadeh, 1965) initiated the idea of fuzzy sets. The rapid evolution of fuzzy is simple to handle and makes it possible to investigate the level of uncertainty in nature in a purely formal and mathematical approach. The concept of continuous - t norms defined by Schweizer and Sklar (1960). Kramosil and Michalek (1975) compared and got the results from probabilistic, statistical generalisations of metric spaces by introducing the notion of fuzziness via continuous t- norms to the traditional notion of metric. The one Garbiec (1988) who interpreted the fuzzy concept of Banach contraction

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principle in fuzzy metric spaces. $\alpha - \phi$- fuzzy cone contraction results were proved with integral equation application by S.U Rehman et al. (2021).

Only membership functions are dealt within fuzzy metric spaces. Park (2004) established an intuitionistic fuzzy metric space which is used to distribute with both membership and non-membership functions. The idea of an intuitionistic fuzzy b-metric space was introduced by Konwar (2020), who also demonstrated a number of fixed point theorems. Mutlu and Gurdal (2016), introduced the notion of bipolar metric spaces and proved fixed point theorems. In the recent past many researchers have established various fixed point results using various types of contractions in the setting of bipolar metric spaces (Kishore et al., 2018; Rao et al., 2018; Kishore, Prasad, et al., 2019; Kishore, Rao, et al., 2019; Gürdal et al., 2020; Mutlu et al., 2020; Kishore et al., 2021; Roy & Saha, 2020; Gaba et al., 2021; Roy & Saha, 2021; Roy et al., 2022) . In 2012 Shah et al. (2012a) developed the theme of intuitionistic fuzzy normal subgroups over a non associative rings. Kausar and Waqar (2019) have initiated the concept of non-associative rings by their intuitionistic fuzzy bi-ideals. Furthermore in 2019 Kausar (2019)developed non-associative ordered semigroups by the properties of their fuzzy ideals with thresholds. Moreover Shah et al. (2012b) introduced the notion of intuitionistic fuzzy normal subrings over a non-associative ring.

In this paper, we introduce the notion of intuitionistic fuzzy bipolar metric space and prove fixed point theorems.

2. Preliminaries

In this section, we provide some basic definitions.

Definition 2.1 (Park, 2004) *A* binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called a continuous triangle norm(CTN) if:

1. $\nu * \xi = \xi * \nu, (\forall)\nu, \xi \in [0,1]$;

2. $*$ is continuous;

3. $\nu * 1 = \nu, (\forall)\nu \in [0,1]$;

4. $(\nu * \xi) * \omega = \nu * (\xi * \omega)$, for all $\nu, \xi, \omega \in [0,1]$;

5. If $\nu \leq \omega$ and $\xi \leq \eta$, with $\nu, \xi, \omega, \eta \in [0,1]$, then $\nu * \xi \leq \omega * \eta$.

Definition 2.2 (Park, 2004) A binary operation $\circ: [0,1] \times [0,1] \to [0,1]$ is called a continuous triangle co-norm(CTCN) if:

1. $\nu \circ \xi = \xi \circ \nu$, for all $\nu, \xi \in [0,1]$;

1. $\circ$ is continuous;

2. $\nu \circ 0 = 0$, for all $\nu \in [0,1]$;

3. $(\nu \circ \xi) \circ \omega = \nu \circ (\xi \circ \omega)$, for all $\nu, \xi, \omega \in [0,1]$;

4. If $\nu \leq \omega$ and $\omega \leq \eta$, with $\nu, \xi, \omega, \eta \in [0,1]$, then $\nu \circ \xi \leq \omega \circ \eta$.

Definition 2.3 (Konwar, 2020) Take $\Omega \neq \emptyset$. *Let* $*$ *be a CTN,* $\circ$ be *a* CTCN, $\mathfrak{b} \geq 1$ and Γ, Υ be fuzzy sets on $\Omega \times \Omega \times (0, +\infty)$. *If* $(\Omega, \Gamma, \Upsilon, *, \circ)$ fulfils all $\aleph, \Lambda \in \Omega$ and $\psi, \varphi > 0$*:*

1. $\Gamma(\aleph, \Lambda, \varphi) + \Upsilon(\aleph, \Lambda, \varphi) \leq 1$;
2. $\Gamma(\aleph, \Lambda, \varphi) > 0$;
3. $\Gamma(\aleph, \Lambda, \varphi) = 1 \Leftrightarrow \aleph = \Lambda$;
4. $\Gamma(\aleph, \Lambda, \varphi) = \Gamma(\Lambda, \aleph, \varphi)$;
5. $\Gamma(\aleph, \lambda, \mathfrak{b}(\varphi + \psi)) \geq \Gamma(\aleph, \Lambda, \varphi) * \Gamma(\Lambda, \lambda, \psi)$;
6. $\Gamma(\aleph, \Lambda, \cdot)$ is an increasing function of $\mathbb{R}^+$ and $\lim_{\varphi \to +\infty} \Gamma(\aleph, \Lambda, \varphi) = 1$;
7. $\Upsilon(\aleph, \Lambda, \varphi) > 0$;
8. $\Upsilon(\aleph, \Lambda, \varphi) = 0 \Leftrightarrow \aleph = \Lambda$;
9. $\Upsilon(\aleph, \Lambda, \varphi) = \Upsilon(\Lambda, \aleph, \varphi)$;
10. $\Upsilon(\aleph, \lambda, \mathfrak{b}(\varphi + \psi)) \leq \Upsilon(\aleph, \Lambda, \varphi) \circ \Upsilon(\Lambda, \lambda, \psi)$;
11. $\Upsilon(\aleph, \Lambda, \cdot)$ is a decreasing function of $\mathbb{R}^+$ and $\lim_{\varphi \to +\infty} \Upsilon(\aleph, \Lambda, \varphi) = 0$,

Then, $(\Omega, \Gamma, \Upsilon, *, \circ)$ is an intuitionistic fuzzy $\mathfrak{b}$-metric space.

Definition 2.4 (Mutlu & Gürdal, 2016) Let Ω and Θ *be* non-void sets and $\varrho : \Omega \times \Theta \to [0, +\infty)$ be *a* function*,* such *that*

1. $\varrho(\aleph, \Lambda) = 0$ if and only if $\aleph = \Lambda$, for all $(\aleph, \Lambda) \in \Omega \times \Theta$;
2. $\varrho(\aleph, \Lambda) = \varrho(\aleph, \Lambda)$, for all $(\aleph, \Lambda) \in \Omega \cap \Theta$;
3. $\varrho(\aleph, \Lambda) \leq \varrho(\aleph, \gamma) + \varrho(\aleph_1, \gamma) + \varrho(\aleph_1, \Lambda)$, for all $\aleph, \aleph_1 \in \Omega$ and $\gamma, \Lambda \in \Theta$.

The pair $(\Omega, \Theta, \varrho)$ is called a bipolar metric space.

Definition 2.5 (Shah et al., 2012b) Let Ω and Θ be two non-void sets. We say that quadruple (Ω, Θ, Γ, ∗) fuzzy bipolar metric space if ∗ is continuous ϱ-norm and Γ is a fuzzy set on Ω × Θ × (0, ∞), fulfill the following conditions for all ϱ, ω, r > 0:

1. $\Gamma(\aleph, \Lambda, \varphi) > 0$; for all ((ℵ, Λ) ∈ Ω ×Θ;
2. Γ (ℵ, Λ, φ) = 1 iff ℵ = Λ for ℵ ∈ Ω and Λ ∈ Θ;
3. Γ (ℵ, Λ, φ) = Γ (Λ, ℵ, φ) for all ℵ, Λ ∈ Ω ∩ Θ;
4. Γ (ℵ1, η2, ϱ + ω + r) ≥ Γ (ℵ1, λ1, ϱ) ∗Γ(ℵ2, λ1, ω)∗Γ(ℵ2, λ2, r) for all ℵ1, ℵ2 ∈ Ω and λ1, λ2 ∈ Θ;

5. $\Gamma((\aleph, \Lambda,.) : [0, \infty) \to [0, 1]$ is left continuous;

6. $\Gamma(\aleph, \Lambda,.)$ is non-decreasing for all $\aleph \in \Omega$ and $\Lambda \in \Theta$.

In this section, inspired by the notions of contraction mapping, and bipolar metric space, we introduce a new concept intuitionistic fuzzy bipolar metric space and prove some fixed-point theorems for these contraction mappings in the setting of complete intuitionistic bipolar metric space. Also, we give some examples to illustrate our results. Furthermore, we apply our results to show the existence and uniqueness of a solution of the first-order ordinary differential equation.

3. Fixed point theorems on intuitionistic fuzzy bipolar metric space:

In this section, we present intuitionistic fuzzy bipolar metric space and demonstrate some fixed-point results.

Definition 3.1 Let $\Omega \neq \emptyset$, $\Theta \neq \emptyset$ be two sets and $*$ be a CTN, $\circ$ be a CTCN and Γ, Υ be neutrosophic sets on $\Omega \times \Theta \times (0, +\infty)$ is said to be a intuitionistic fuzzy bipolar metric on $\Omega \times \Theta$, if for all $\aleph, \varpi \in \Omega$, $\Lambda, \lambda \in \Theta$ and $\varphi, \hat{s}, \widehat{w} > 0$, the following conditions are satisfied:

1. $\Gamma(\aleph, \Lambda, \varphi) + \Upsilon(\aleph, \Lambda, \varphi) \leq 1$;

2. $\Gamma(\aleph, \Lambda, \varphi) > 0$;

3. $\Gamma(\aleph, \Lambda, \varphi) = 1$ for all $\varphi > 0$, if and only if $\aleph = \Lambda$;

4. $\Gamma(\aleph, \Lambda, \varphi) = \Gamma(\Lambda, \aleph, \varphi)$;

5. $\Gamma(\aleph, \lambda, \varphi + \psi + \zeta) \geq \Gamma(\aleph, \Lambda, \varphi) * \Gamma(\varpi, \Lambda, \psi) * \Gamma(\varpi, \lambda, \zeta)$;

6. $\Gamma(\aleph, \Lambda, \cdot): (0, +\infty) \to [0,1]$ is continuous and $\lim_{\varphi \to +\infty} \Gamma(\aleph, \Lambda, \varphi) = 1$;

7. $\Gamma(\aleph, \Lambda, \cdot)$ is increasing function;

8. $\Upsilon(\aleph, \Lambda, \varphi) < 1$;

9. $\Upsilon(\aleph, \Lambda, \varphi) = 0$ for all $\varphi > 0$, if and only if $\aleph = \Lambda$;

10. $\Upsilon(\aleph, \Lambda, \varphi) = \Upsilon(\Lambda, \aleph, \varphi)$;

11. $\Upsilon(\aleph, \lambda, \varphi + \psi + \zeta) \leq \Upsilon(\aleph, \Lambda, \varphi) \circ \Upsilon(\varpi, \Lambda, \psi) \circ \Upsilon(\varpi, \lambda, \zeta)$;

12. $\Upsilon(\aleph, \Lambda, \cdot): (0, +\infty) \to [0,1]$ is continuous and $\lim_{\varphi \to +\infty} \Upsilon(\aleph, \Lambda, \varphi) = 0$;

13. $\Upsilon(\aleph, \Lambda, \cdot)$ is decreasing function;

14. If $\varphi \leq 0$, then $\Gamma(\aleph, \Lambda, \varphi) = 0$ and $\Upsilon(\aleph, \Lambda, \varphi) = 1$.

Then, $(\Omega,\Theta,\Gamma,\Upsilon,*,\circ)$ is called a intuitionistic fuzzy bipolar metric space.

Example 3.1 *Let* $\Omega=\{1,5,3,7\}$ *and* $\Theta=\{1,2,6,4\}$. *Define* $\Gamma,\Upsilon:\Omega\times\Theta\times(0,+\infty)\to[0,1]$ *as*

$$\Gamma(\aleph,\Lambda,\varphi)=\begin{cases}1, & \text{if} \quad \aleph=\Lambda\\ \dfrac{\varphi}{\varphi+\max\{\aleph,\Lambda\}}, & \text{if} \quad \text{otherwise,}\end{cases}$$

and

$$\Upsilon(\aleph,\Lambda,\varphi)=\begin{cases}0, & \text{if} \quad \aleph=\Lambda\\ \dfrac{\max\{\aleph,\Lambda\}}{\varphi+\max\{\aleph,\Lambda\}}, & \text{if} \quad \text{otherwise.}\end{cases}$$

Then, $(\Omega,\Gamma,\Upsilon,*,\circ)$ is an intuitionistic fuzzy bipolar metric space with CTN, $\nu*\xi=\nu\xi$ and CTCN,
$\nu\circ\bar{a}=\max\{\nu,\bar{a}\}$.
Proof. Here, we prove 3.1 and 3.1, others are obvious.

Let $\aleph=1,\Lambda=2,\ \varpi=3$ and $\lambda=4$. Then

$$\Gamma(1,4,\varphi+\psi+\zeta)=\frac{\varphi+\psi+\zeta}{\varphi+\psi+\zeta+\max\{1,4\}}=\frac{\varphi+\psi+\zeta}{\varphi+\psi+\zeta+4}.$$

On the other hand,

$$\Gamma(1,2,\varphi)=\frac{\varphi}{\varphi+\max\{1,2\}}=\frac{\varphi}{\varphi+2}=\frac{\varphi}{\varphi+2},$$

$$\Gamma(2,3,\psi)=\frac{\psi}{\psi+\max\{2,3\}}=\frac{\psi}{\psi+3}=\frac{\psi}{\psi+3}$$

and

$$\Gamma(3,4,\zeta)=\frac{\zeta}{\zeta+\max\{3,4\}}=\frac{\zeta}{\zeta+4}=\frac{\zeta}{\zeta+4}.$$

Therefore,

$$\frac{\varphi+\psi+\zeta}{\varphi+\psi+\zeta+3}\geq\frac{\varphi}{\varphi+2}\cdot\frac{\psi}{\psi+3}\cdot\frac{\zeta}{\zeta+4}.$$

Hence,

$$\Gamma(\aleph,\lambda,\varphi+\psi+\zeta)\geq\Gamma(\aleph,\Lambda,\varphi)*\Gamma(\Lambda,\varpi,\psi)*\Gamma(\varpi,\lambda,\zeta),\qquad\forall\ \varphi,\psi,\zeta>0.$$

Now,

$$\Upsilon(1,4,\varphi+\psi+\zeta)=\frac{\max\{1,4\}}{\varphi+\psi+\zeta+\max\{1,4\}}=\frac{4}{\varphi+\psi+\zeta+4}.$$

On the other hand,

$$\Upsilon(1,2,\varphi) = \frac{\max\{1,2\}}{\varphi + \max\{1,2\}} = \frac{2}{\varphi + 2} = \frac{2}{\varphi + 2}$$

$$\Upsilon(2,3,\psi) = \frac{\max\{2,3\}}{\psi + \max\{2,3\}} = \frac{3}{\psi + 3} = \frac{3}{\psi + 3}$$

and

$$\Upsilon(3,4,\zeta) = \frac{\max\{3,4\}}{\zeta + \max\{3,4\}} = \frac{4}{\zeta + 4} = \frac{4}{\zeta + 4}.$$

That is,

$$\frac{4}{\varphi + \psi + \zeta + 4} \leq \max\{\frac{2}{\varphi + 2}, \frac{3}{\psi + 3}, \frac{4}{\zeta + 4}\}.$$

Therefore,

$$\Upsilon(\aleph, \lambda, \varphi + \psi + \zeta) \leq \Upsilon(\aleph, \Lambda, \varphi) \circ \Upsilon(\varpi, \lambda, \psi) \circ \Upsilon(\varpi, \lambda, \zeta), \qquad \forall\ \varphi, \psi, \zeta > 0.$$

Hence, $(\Omega, \Gamma, \Upsilon, *, \circ)$ is an intuitionistic fuzzy bipolar metric space.

Definition 3.2 Let $\mathfrak{p}: \Omega_1 \cup \Theta_1 \to \Omega_2 \cup \Theta_2$ be *a* mapping, where (Ω_1, Θ_1) and (Ω_2, Θ_2) are pairs of sets *(H)*

1. If $\mathfrak{p}(\Omega_1) \subseteq \Omega_2$ and $\mathfrak{p}(\Theta_1) \subseteq \Theta_2$, then $\mathfrak{p}$ is called a covariant map, or a map from $(\Omega_1, \Theta_1, \Gamma_1, \Upsilon_1, *, \circ)$ to $(\Omega_2, \Theta_2, \Gamma_2, \Upsilon_2, *, \circ)$ and this is written as $\mathfrak{p}: (\Omega_1, \Theta_1, \Gamma_1, \Upsilon_1, *, \circ) \rightrightarrows (\Omega_2, \Theta_2, \Gamma_2, \Upsilon_2, *, \circ)$.

2. If $\mathfrak{p}(\Omega_1) \subseteq \Theta_2$ and $\mathfrak{p}(\Theta_1) \subseteq \Omega_2$, then $\mathfrak{p}$ is called a contravariant map from $(\Omega_1, \Theta_1, \Gamma_1, \Upsilon_1, *, \circ)$ to $(\Omega_2, \Theta_2, \Gamma_2, \Upsilon_2, *, \circ)$ and this is denoted as $\mathfrak{p}: (\Omega_1, \Theta_1, \Gamma_1, \Upsilon_1, *, \circ) \leftrightarrows (\Omega_2, \Theta_2, \Gamma_2, \Upsilon_2, *, \circ)$.

Example 3.2 If $\Omega_1 \cup \Theta_1 = X = \{0, 1\}$, then $p(X) = \{\{\}, \{0\}, \{1\}, X\}$. Supposet $\Gamma(0)=\{\}$ and $\Gamma(1) = X$. Then $\mathfrak{p}(\Gamma)$ is the functon which sends any subset U of X to its image $\Gamma(U)$, which in this case means $\{\} \to \Gamma(\{\}) = \{\}$, where $\to$ denotes the mapping under $\mathfrak{p}(\Gamma)$, so this could also be written as $(p(\Gamma))(\{\}) = \{\}$. For the other values, $\{0\} \to \Gamma(\{0\}) = \{\Gamma(0)\} = \{\{\}\}, \{1\} \to \Gamma(\{1\}) = \{\Gamma(1)\} = \{X\}, \{0, 1\} \to \Gamma(\{0, 1\}) = \{\Gamma(0), \Gamma(1)\} = \{\{\}, X\}$.

Definition 3.3 Let $(\Omega, \Theta, \Gamma, \Upsilon, *, \circ)$ is a intuitionistic fuzzy bipolar metric space.

1. A point $\aleph \in \Omega \cup \Theta$ is said to be a left point if $\aleph \in \Omega$, a right point if $\aleph \in \Theta$ and a central point if both hold.
2. A sequence $\{\aleph_\alpha\} \subset \Omega$ is called a left sequence and a sequence $\{\beta_n\} \subset \Theta$ is called a right sequence.
3. A sequence $\{\aleph_\alpha\} \subset \Omega \cup \Theta$ is said to converge to a point $\aleph$ if and only if $\{\aleph_\alpha\}$ is a left sequence, $\aleph$ is a right point and

$$\lim_{\alpha \to +\infty} \Gamma(\aleph_\alpha, \aleph, \varphi) = 1, \lim_{\alpha \to +\infty} \Upsilon(\aleph_\alpha, \aleph, \varphi) = 0, \quad \text{forall} \quad \varphi > 0$$

or $\{\aleph_\alpha\}$ is a right sequence, $\aleph$ is a left point and

$$\lim_{\alpha\to+\infty}\Gamma(\aleph,\aleph_\alpha,\varphi)=1,\ \lim_{\alpha\to+\infty}\Upsilon(\aleph,\aleph_\alpha,\varphi)=0,\quad \text{forall}\quad \varphi>0.$$

4. A sequence $\{(\aleph_\alpha,\beta_\alpha)\}\subset\Omega\times\Theta$ is called a bisequence. If the sequences $\{\aleph_\alpha\}$ and $\{\beta_\alpha\}$ are both converge, then the bisequence $\{(\aleph_\alpha,\beta_\alpha)\}$ is called convergent in $\Omega\times\Theta$.

5. If $\{\aleph_\alpha\}$ and $\{\beta_\alpha\}$ are both converge to a point $\beta\in\Omega\cap\Theta$, then the bisequence $\{(\aleph_\alpha,\beta_\alpha)\}$ is called biconvergent. A sequence $\{(\aleph_\alpha,\beta_\alpha)\}$ is a Cauchy bisequence if

$$\lim_{\alpha,\kappa\to+\infty}\Gamma(\aleph_\alpha,\beta_\kappa,\varphi)=1,\ \lim_{\alpha,\kappa\to+\infty}\Upsilon(\aleph_\alpha,\beta_\kappa,\varphi)=0,\ \text{for all}\ \varphi>0.$$

6. A intuitionistic fuzzy bipolar metric space is said to be complete if every Cauchy bisequence is convergent.

Example 3.3 Let $\Omega=(1,\infty)$ and $\Theta=[-1,1]$. Define $\Gamma:\Omega\times\Theta\to\mathbb{R}^+$ as $\Gamma(\aleph,\Lambda)=|\aleph^2-\Lambda^2|$. Then $(\Omega,\Theta,\Gamma,\Upsilon,*,\circ)$ is a intuitionistic fuzzy bipolar metric space. Note that the left sequence $(1+\frac{1}{\alpha})$ converges to right points 1 and −1.

Example 3.4 Let $\Omega=(0,1)$, $\Theta=(2,3)$. Consider the subspace (Ω,Θ,Γ) of the space $(\mathbb{R},\mathbb{R},\Gamma)$ where Γ is the usual metric on $\mathbb{R}$. Since $\Gamma(\aleph,\Lambda)>1$ for any $(\aleph,\Lambda)\in\Omega$ x Θ, there is no Cauchy bisequence in (Ω,Θ,Γ). Thus, it is vacuously true that (Ω,Θ,Γ) is complete. However, note that $\Gamma\Omega$ is equal to the usual metric on $\Omega=(0,1)$ and $(\Omega,\Gamma\Omega)$ is not complete, hence (Ω,Θ,Γ) is not bi complete.

Lemma 3.1 Let $\{\aleph_\alpha\}$ be a Cauchy sequence in intuitionistic fuzzy bipolar metric space $(\Omega,\Theta,\Gamma,\Upsilon,*,\circ)$ such that $\aleph_\alpha\neq\aleph_\kappa$ whenever $\kappa,\alpha\in\mathbb{N}$ with $\alpha\neq\kappa$. Then the sequence $\{\aleph_\alpha\}$ can converge to, at most, one limit point.

Proof. Assume that $\aleph_\alpha\to\aleph\in\Theta$ and $\aleph_\alpha\to\Lambda\in\Omega\cap\Theta$, for $\aleph\neq\Lambda$. Then, $\lim_{\alpha\to+\infty}\Gamma(\aleph_\alpha,\aleph,\varphi)=1,\lim_{\alpha\to+\infty}\Upsilon(\aleph_\alpha,\aleph,\varphi)=0$ and $\lim_{\alpha\to+\infty}\Gamma(\aleph_\alpha,\Lambda,\varphi)=1,\lim_{\alpha\to+\infty}\Upsilon(\aleph_\alpha,\Lambda,\varphi)=0$, for all $\varphi>0$. Suppose

$$\Gamma(\aleph,\Lambda,\varphi)\geq\Gamma(\aleph,\aleph_\alpha,\frac{\varphi}{3})*\Gamma(\aleph_\alpha,\aleph_{\alpha+1},\frac{\varphi}{3})*\Gamma(\aleph_{\alpha+1},\Lambda,\frac{\varphi}{3})$$

$$\to 1*1*1,\quad \text{as}\quad \alpha\to+\infty,$$

$$\Upsilon(\aleph,\Lambda,\varphi)\leq\Upsilon(\aleph,\aleph_\alpha,\frac{\varphi}{3})\circ\Upsilon(\aleph_\alpha,\aleph_{\alpha+1},\frac{\varphi}{3})\circ\Upsilon(\aleph_{\alpha+1},\Lambda,\frac{\varphi}{3})$$

$$\to 0\circ 0\circ 0,\quad \text{as}\quad \alpha\to+\infty.$$

That is $\Gamma(\aleph,\Lambda,\varphi)\geq 1*1*1=1,\Upsilon(\aleph,\Lambda,\varphi)\leq 0\circ 0\circ 0=0$ and $\mathcal{B}(\aleph,\Lambda,\varphi)\leq 0\circ 0\circ 0=0$. Hence $\aleph=\Lambda$.

Lemma 3.2 Let $(\Omega,\Theta,\Gamma,\Upsilon,*,\circ)$ is a intuitionistic fuzzy bipolar metric space. If for some $0<\theta<1$ and for any $\aleph,\Lambda\in\Omega,\varphi>0$,

$$\Gamma(\aleph,\Lambda,\varphi)\geq\Gamma(\aleph,\Lambda,\frac{\varphi}{\theta}),\Upsilon(\aleph,\Lambda,\varphi)\leq\Upsilon(\aleph,\Lambda,\frac{\varphi}{\theta}),\tag{1}$$

then $\aleph=\Lambda$.

Proof. (1) implies that

$$\Gamma(\aleph,\Lambda,\varphi)\geq\Gamma(\aleph,\Lambda,\frac{\varphi}{\theta^\alpha}),\Upsilon(\aleph,\Lambda,\varphi)\leq\Upsilon(\aleph,\Lambda,\frac{\varphi}{\theta^\alpha}),\alpha\in\mathbb{N},\varphi>0.$$

Now

$$\Gamma(\aleph,\Lambda,\varphi) \geq \lim_{\alpha\to+\infty} \Gamma(\aleph,\Lambda,\frac{\varphi}{\theta^{\alpha}}) = 1,$$

$$\Upsilon(\aleph,\Lambda,\varphi) \leq \lim_{\alpha\to+\infty} \Upsilon(\aleph,\Lambda,\frac{\varphi}{\theta^{\alpha}}) = 0.$$

Also, by definition of 3.1 and 3.1 that is, $\aleph = \Lambda$.

Theorem 3.1 Suppose $(\Omega,\Theta,\Gamma,\Upsilon,*,\circ)$ is a complete intuitionistic fuzzy bipolar metric space with $0 < \theta < 1$. Let $\mathfrak{p}: \Omega \cup \Theta \to \Omega \cup \Theta$ be a mapping satisfying:

1. $\mathfrak{p}(\Omega) \subseteq \Omega$ and $\mathfrak{p}(\Theta) \subseteq \Theta$;

2. $\Gamma(\mathfrak{p}\aleph,\mathfrak{p}\Lambda,\theta\varphi) \geq \Gamma(\aleph,\Lambda,\varphi)$, (2)

and $\Upsilon(\mathfrak{p}\aleph,\mathfrak{p}\Lambda,\theta\varphi) \leq \Upsilon(\aleph,\Lambda,\varphi)$ (3)

for all $\aleph \in \Omega$, $\Lambda \in \Theta$ and $\varphi > 0$.

Then $\mathfrak{p}$ has a unique fixed point.

Proof. Let $\aleph_0 \in \Omega$ and $\Lambda_0 \in \Theta$ and assume that $\mathfrak{p}(\aleph_\alpha) = \aleph_{\alpha+1}$ and $\mathfrak{p}(\Lambda_\alpha) = \Lambda_{\alpha+1}$ for all $\alpha \in \mathbb{N} \cup \{0\}$. Then we get $(\aleph_\alpha, \Theta_\alpha)$as a bisequence on intuitionistic fuzzy bipolar metric space$(\Omega,\Theta,\Gamma,\Upsilon,*,\circ)$. Now, we have

$$\Gamma(\aleph_1,\Lambda_1,\varphi) = \Gamma(\mathfrak{p}\aleph_0,\mathfrak{p}\Lambda_0,\varphi) \geq \Gamma(\aleph_0,\Lambda_0,\frac{\varphi}{\theta}),$$

and

$$\Upsilon(\aleph_1,\Lambda_1,\varphi) = \Upsilon(\mathfrak{p}\aleph_0,\mathfrak{p}\Lambda_0,\varphi) \leq \Upsilon(\aleph_0,\Lambda_0,\frac{\varphi}{\theta}),$$

for all $\varphi > 0$ and $\alpha \in \mathbb{N}$. By simple induction, we get

$$\Gamma(\aleph_\alpha,\Lambda_\alpha,\varphi) = \Gamma(\mathfrak{p}\aleph_{\alpha-1},\mathfrak{p}\Lambda_{\alpha-1},\varphi) \geq \Gamma(\aleph_{\alpha-1},\Lambda_{\alpha-1},\frac{\varphi}{\theta}) \geq \Gamma(\aleph_{\alpha-2},\Lambda_{\alpha-2},\frac{\varphi}{\theta^2})$$
$$\geq \Gamma(\aleph_{\alpha-3},\Lambda_{\alpha-3},\frac{\varphi}{\theta^3}) \geq \cdots \geq \Gamma(\aleph_0,\Lambda_0,\frac{\varphi}{\theta^\alpha}),$$

and

$$\Upsilon(\aleph_\alpha,\Lambda_\alpha,\varphi) = \Upsilon(\mathfrak{p}\aleph_{\alpha-1},\mathfrak{p}\Lambda_{\alpha-1},\varphi) \leq \Upsilon(\aleph_{\alpha-1},\Lambda_{\alpha-1},\frac{\varphi}{\theta}) \leq \Upsilon(\aleph_{\alpha-2},\Lambda_{\alpha-2},\frac{\varphi}{\theta^2})$$
$$\leq \Upsilon(\aleph_{\alpha-3},\Lambda_{\alpha-3},\frac{\varphi}{\theta^3}) \leq \cdots \leq \Upsilon(\aleph_0,\Lambda_0,\frac{\varphi}{\theta^\alpha}).$$

We obtain

$$\Gamma(\aleph_\alpha,\Lambda_\alpha,\varphi) \geq \Gamma(\aleph_0,\Lambda_0,\tfrac{\varphi}{\theta^\alpha}), \Upsilon(\aleph_\alpha,\Lambda_\alpha,\varphi) \leq \Upsilon(\aleph_0,\Lambda_0,\tfrac{\varphi}{\theta^\alpha}) \quad (4)$$

and

$$\Gamma(\aleph_{\alpha+1},\Lambda_\alpha,\varphi)\geq\Gamma(\aleph_1,\Lambda_0,\tfrac{\varphi}{\theta^\alpha}),\Upsilon(\aleph_{\alpha+1},\Lambda_\alpha,\varphi)\leq\Upsilon(\aleph_1,\Lambda_0,\tfrac{\varphi}{\theta^\alpha}). \quad (5)$$

Letting $\alpha<\kappa$, for $\alpha,\kappa\in\mathbb{N}$. Then,

$$\Gamma(\aleph_\alpha,\Lambda_\kappa,\varphi)\geq\Gamma(\aleph_\alpha,\Lambda_\alpha,\frac{\varphi}{3})*\Gamma(\aleph_{\alpha+1},\Lambda_\alpha,\frac{\varphi}{3})*\Gamma(\aleph_{\alpha+1},\Lambda_\kappa,\frac{\varphi}{3})$$

$$\vdots$$

$$\geq\Gamma(\aleph_\alpha,\Lambda_\alpha,\frac{\varphi}{3})*\Gamma(\aleph_{\alpha+1},\Lambda_\alpha,\frac{\varphi}{3})*\cdots*\Gamma(\aleph_{\kappa-1},\Lambda_{\kappa-1},\frac{\varphi}{3^{\kappa-1}})*\Gamma(\aleph_\kappa,\Lambda_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}*\Gamma(\aleph_\kappa,\Lambda_\kappa,\frac{\varphi}{3^{\kappa-1}}),$$

and

$$\Upsilon(\aleph_\alpha,\Lambda_\kappa,\varphi)\leq\Upsilon(\aleph_\alpha,\Lambda_\alpha,\frac{\varphi}{3})\circ\Upsilon(\aleph_{\alpha+1},\Lambda_\alpha,\frac{\varphi}{3})\circ\Upsilon(\aleph_{\alpha+1},\Lambda_\kappa,\frac{\varphi}{3})$$

$$\vdots$$

$$\leq\Upsilon(\aleph_\alpha,\Lambda_\alpha,\frac{\varphi}{3})\circ\Upsilon(\aleph_{\alpha+1},\Lambda_\alpha,\frac{\varphi}{3})\circ\cdots\circ\Upsilon(\aleph_{\kappa-1},\Lambda_{\kappa-1},\frac{\varphi}{3^{\kappa-1}})\circ\Upsilon(\aleph_\kappa,\Lambda_{\kappa-1},\frac{\varphi}{3^{\kappa-1}})\circ\Upsilon(\aleph_\kappa,\Lambda_\kappa,\frac{\varphi}{3^{\kappa-1}}).$$

Therefore,

$$\begin{aligned}\Gamma(\aleph_\alpha,\Lambda_\kappa,\varphi)&\geq\Gamma(\aleph_\alpha,\Lambda_\alpha,\frac{\varphi}{3})*\Gamma(\aleph_{\alpha+1},\Lambda_\alpha,\frac{\varphi}{3})*\cdots*\Gamma(\aleph_{\kappa-1},\Lambda_{\kappa-1},\frac{\varphi}{3^{\kappa-1}})\\&*\Gamma(\aleph_\kappa,\Lambda_{\kappa-1},\frac{\varphi}{3^{\kappa-1}})*\Gamma(\aleph_\kappa,\Lambda_\kappa,\frac{\varphi}{3^{\kappa-1}})\\&\geq\Gamma(\aleph_0,\Lambda_0,\frac{\varphi}{3\theta^\alpha})*\Gamma(\aleph_1,\Lambda_0,\frac{\varphi}{3\theta^\alpha})*\cdots*\Gamma(\aleph_0,\Lambda_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}})\\&*\Gamma(\aleph_1,\Lambda_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}})*\Gamma(\aleph_0,\Lambda_0,\frac{\varphi}{3^{\kappa-1}\theta^\kappa}),\end{aligned}$$

and

$\Upsilon(\aleph_\alpha,\Lambda_\kappa,\varphi)\leq\Upsilon(\aleph_\alpha,\Lambda_\alpha,\frac{\varphi}{3})\circ\Upsilon(\aleph_{\alpha+1},\Lambda_\alpha,\frac{\varphi}{3})\circ\cdots\circ\Upsilon(\aleph_{\kappa-1},\Lambda_{\kappa-1},\frac{\varphi}{3^{\kappa-1}})\circ$
$\Upsilon(\aleph_\kappa,\Lambda_{\kappa-1},\frac{\varphi}{3^{\kappa-1}})\circ\Upsilon(\aleph_\kappa,\Lambda_\kappa,\frac{\varphi}{3^{\kappa-1}})\leq\Upsilon\left(\aleph_0,\Lambda_0,\frac{\varphi}{3\theta^\alpha}\right)\circ\Upsilon\left(\aleph_1,\Lambda_0,\frac{\varphi}{3\theta^\alpha}\right)\circ\cdots\circ$
$\Upsilon\left(\aleph_0,\Lambda_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right)\Upsilon(\aleph_1,\Lambda_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}})\circ\Upsilon(\aleph_0,\Lambda_0,\frac{\varphi}{3^{\kappa-1}\theta^\kappa}).$

Which implies that,

$$\begin{aligned}\Gamma(\aleph_\alpha,\Lambda_\kappa,\varphi)&\geq\Gamma(\aleph_0,\Lambda_0,\frac{\varphi}{3\theta^\alpha})*\Gamma(\aleph_1,\Lambda_0,\frac{\varphi}{3\theta^\alpha})*\cdots*\Gamma(\aleph_0,\Lambda_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}})\\&*\Gamma(\aleph_1,\Lambda_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}*\Gamma(\aleph_0,\Lambda_0,\frac{\varphi}{3^{\kappa-1}\theta^\kappa}),\end{aligned}$$

$$\begin{aligned}\Upsilon(\aleph_\alpha,\Lambda_\kappa,\varphi)&\leq\Upsilon(\aleph_0,\Lambda_0,\frac{\varphi}{3\theta^\alpha})\circ\Upsilon(\aleph_1,\Lambda_0,\frac{\varphi}{3\theta^\alpha})\circ\cdots\circ\Upsilon(\aleph_0,\Lambda_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\\&\circ\Upsilon(\aleph_1,\Lambda_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}})\circ\Upsilon(\aleph_0,\Lambda_0,\frac{\varphi}{3^{\kappa-1}\theta^\kappa}).\end{aligned}$$

As $\alpha, \kappa \to +\infty$, we deduce

$$\lim_{\alpha,\kappa\to+\infty} \Gamma(\aleph_\alpha, \Lambda_\kappa, \varphi) = 1 * 1 * \cdots * 1 = 1,$$

$$\text{and} \lim_{\alpha,\kappa\to+\infty} \Upsilon(\aleph_\alpha, \Lambda_\kappa, \varphi) = 0 \circ 0 \circ \cdots \circ 0 = 0.$$

Which implies that bisequence $(\aleph_\alpha, \Lambda_\alpha)$ is a Cauchy bisequence. Since $(\Omega, \Theta, \Gamma, \Upsilon, *, \circ)$ is a complete intuitionistic fuzzy bipolar metric space. Then, $\{\aleph_\alpha\} \to \aleph$ and $\{\Lambda_\alpha\} \to \aleph$, where $\aleph \in \Omega \cap \Theta$. Using 3.1 and 3.1, we get

$$\begin{aligned} \Gamma(\aleph, \mathfrak{p}\aleph, \varphi) &\geq \Gamma(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}) * \Gamma(\aleph_{\alpha+1}, \aleph_{\alpha+1}, \frac{\varphi}{3}) * \Gamma(\aleph_{\alpha+1}, \mathfrak{p}\aleph, \frac{\varphi}{3} \\ &= \Gamma(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}) * \Gamma(\mathfrak{p}\aleph_\alpha, \mathfrak{p}\aleph_\alpha, \frac{\varphi}{3}) * \Gamma(\mathfrak{p}\aleph_\alpha, \mathfrak{p}\aleph, \frac{\varphi}{3}) \\ &\to 1 * 1 * 1 = 1 \quad \text{as} \quad \alpha \to +\infty, \end{aligned}$$

$$\begin{aligned} \Upsilon(\aleph, \mathfrak{p}\aleph, \varphi) &\leq \Upsilon(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \mathfrak{p}\aleph, \frac{\varphi}{3} \\ &= \Upsilon(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \Upsilon(\mathfrak{p}\aleph_{\alpha+1}, \mathfrak{p}\aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \Upsilon(\mathfrak{p}\aleph_{\alpha+1}, \mathfrak{p}\aleph, \frac{\varphi}{3}) \\ &\to 0 \circ 0 \circ 0 = 0 \quad \text{as} \quad \alpha \to +\infty. \end{aligned}$$

Hence, $\mathfrak{p}\aleph = \aleph$.

Now, we examine the uniqueness. Let $\mathfrak{p}\omega = \omega$ for some $\omega \in \Omega \cap \Theta$, then

$$1 \geq \Gamma(\omega, \aleph, \varphi) = \Gamma(\mathfrak{p}\omega, \mathfrak{p}\aleph, \varphi) \geq \Gamma(\omega, \aleph, \frac{\varphi}{\theta}) = \Gamma(\mathfrak{p}\omega, \mathfrak{p}\aleph, \frac{\varphi}{\theta})$$

$$\geq \Gamma(\omega, \aleph, \frac{\varphi}{\theta^2}) \geq \cdots \geq \Gamma(\omega, \aleph, \frac{\varphi}{\theta^\alpha}) \to 1 \quad \text{as} \quad \alpha \to +\infty,$$

$$0 \leq \Upsilon(\omega, \aleph, \varphi) = \Upsilon(\mathfrak{p}\omega, \mathfrak{p}\aleph, \varphi) \leq \Upsilon(\omega, \aleph, \frac{\varphi}{\theta}) = \Upsilon(\mathfrak{p}\omega, \mathfrak{p}\aleph, \frac{\varphi}{\theta})$$

$$\leq \Upsilon(\omega, \aleph, \frac{\varphi}{\theta^2}) \leq \cdots \leq \Upsilon(\omega, \aleph, \frac{\varphi}{\theta^\alpha}) \to 0 \quad \text{as} \quad \alpha \to +\infty,$$

by using 3.1 and 3.1, $\aleph = \omega$.

Theorem 3.2 Suppose $(\Omega, \Theta, \Gamma, \Upsilon, *, \circ)$ is a complete intuitionistic fuzzy bipolar metric space with $0 < \theta < 1$. Let $\mathfrak{p}: \Omega \cup \Theta \to \Omega \cup \Theta$ be a mapping satisfying:

1. $\mathfrak{p}(\Omega) \subseteq \Theta$ and $\mathfrak{p}(\Theta) \subseteq \Omega$;

2. $\Gamma(\mathfrak{p}\aleph, \mathfrak{p}\Lambda, \theta\varphi) \geq \Gamma(\Lambda, \aleph, \varphi)$,

and $\Upsilon(\mathfrak{p}\aleph, \mathfrak{p}\Lambda, \theta\varphi) \leq \Upsilon(\Lambda, \aleph, \varphi)$, (6)

for all $\aleph \in \Omega$, $\Lambda \in \Theta$ and $\varphi > 0$.

Then $\mathfrak{p}$ has a unique fixed point.

Proof. Let $\aleph_0 \in \Omega$ and $\Lambda_0 \in \Theta$ and assume that $\mathfrak{p}(\aleph_\alpha) = \Lambda_\alpha$ and $\mathfrak{p}(\Lambda_\alpha) = \aleph_{\alpha+1}$ for all $\alpha \in \mathbb{N} \cup \{0\}$. Then we get $(\aleph_\alpha, \Theta_\alpha)$ as a bisequence on intuitionistic fuzzy bipolar metric space $(\Omega, \Theta, \Gamma, \Upsilon, *, \circ)$. Now, we have

$$\Gamma(\aleph_1, \Lambda_0, \varphi) = \Gamma(\mathfrak{p}\Lambda_0, \mathfrak{p}\aleph_0, \varphi) \geq \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{\theta}),$$

and

$$\Upsilon(\aleph_1, \Lambda_0, \varphi) = \Upsilon(\mathfrak{p}\Lambda_0, \mathfrak{p}\aleph_0, \varphi) \leq \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{\theta}),$$

for all $\varphi > 0$ and $\alpha \in \mathbb{N}$. By simple induction, we get

$$\Gamma(\aleph_\alpha, \Lambda_\alpha, \varphi) = \Gamma(\mathfrak{p}\Lambda_{\alpha-1}, \mathfrak{p}\aleph_\alpha, \varphi) \geq \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{\theta^{2\alpha}}),$$

$$\Upsilon(\aleph_\alpha, \Lambda_\alpha, \varphi) = \Upsilon(\mathfrak{p}\Lambda_{\alpha-1}, \mathfrak{p}\aleph_\alpha, \varphi) \leq \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{\theta^{2\alpha}}),$$

and

$$\Gamma(\aleph_{\alpha+1}, \Lambda_\alpha, \varphi) = \Gamma(\mathfrak{p}\Lambda_\alpha, \mathfrak{p}\aleph_\alpha, \varphi) \geq \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{\theta^{2\alpha+1}}),$$

$$\Upsilon(\aleph_{\alpha+1}, \Lambda_\alpha, \varphi) = \Upsilon(\mathfrak{p}\Lambda_\alpha, \mathfrak{p}\aleph_\alpha, \varphi) \leq \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{\theta^{2\alpha+1}}).$$

Letting $\alpha < \kappa$, for $\alpha, \kappa \in \mathbb{N}$. Then,

$$\Gamma(\aleph_\alpha, \Lambda_\kappa, \varphi) \geq \Gamma(\aleph_\alpha, \Lambda_\alpha, \frac{\varphi}{3}) * \Gamma(\aleph_{\alpha+1}, \Lambda_\alpha, \frac{\varphi}{3}) * \Gamma(\aleph_{\alpha+1}, \Lambda_\kappa, \frac{\varphi}{3})$$

$$\vdots$$

$$\geq \Gamma(\aleph_\alpha, \Lambda_\alpha, \frac{\varphi}{3}) * \Gamma(\aleph_{\alpha+1}, \Lambda_\alpha, \frac{\varphi}{3}) * \cdots * \Gamma(\aleph_{\kappa-1}, \Lambda_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}) * \Gamma(\aleph_\kappa, \Lambda_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}} * \Gamma(\aleph_\kappa, \Lambda_\kappa, \frac{\varphi}{3^{\kappa-1}}),$$

and

$$\Upsilon(\aleph_\alpha, \Lambda_\kappa, \varphi) \leq \Upsilon(\aleph_\alpha, \Lambda_\alpha, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \Lambda_\alpha, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \Lambda_\kappa, \frac{\varphi}{3})$$

$$\vdots$$

$$\leq \Upsilon(\aleph_\alpha, \Lambda_\alpha, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \Lambda_\alpha, \frac{\varphi}{3}) \circ \cdots \circ \Upsilon(\aleph_{\kappa-1}, \Lambda_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}) \circ \Upsilon(\aleph_\kappa, \Lambda_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}} \circ \Upsilon(\aleph_\kappa, \Lambda_\kappa, \frac{\varphi}{3^{\kappa-1}}).$$

Therefore,

$$\Gamma(\aleph_\alpha, \Lambda_\kappa, \varphi) \geq \Gamma(\aleph_\alpha, \Lambda_\alpha, \frac{\varphi}{3}) * \Gamma(\aleph_{\alpha+1}, \Lambda_\alpha, \frac{\varphi}{3}) * \cdots * \Gamma(\aleph_{\kappa-1}, \Lambda_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}$$
$$* \Gamma(\aleph_\kappa, \Lambda_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}) * \Gamma(\aleph_\kappa, \Lambda_\kappa, \frac{\varphi}{3^{\kappa-1}})$$
$$\geq \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{3\theta^{2\alpha}}) * \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{3\theta^{2\alpha+1}}) * \cdots * \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{3^{\kappa-1}\theta^{2\kappa-2}})$$
$$* \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{3^{\kappa-1}\theta^{2\kappa-1}}) * \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{3^{\kappa-1}\theta^{2\kappa}}), \Upsilon(\aleph_\alpha, \Lambda_\kappa, \varphi)$$
$$\leq \Upsilon(\aleph_\alpha, \Lambda_\alpha, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \Lambda_\alpha, \frac{\varphi}{3}) \circ \cdots \circ \Upsilon(\aleph_{\kappa-1}, \Lambda_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}})$$
$$\circ \Upsilon(\aleph_\kappa, \Lambda_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}) \circ \Upsilon(\aleph_\kappa, \Lambda_\kappa, \frac{\varphi}{3^{\kappa-1}})$$
$$\leq \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{3\theta^{2\alpha}}) \circ \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{3\theta^{2\alpha+1}}) \circ \cdots \circ \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{3^{\kappa-1}\theta^{2\kappa-2}})$$
$$\circ \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{3^{\kappa-1}\theta^{2\kappa-1}}) \circ \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{3^{\kappa-1}\theta^{2\kappa}}).$$

Which implies that,

$$\Gamma(\aleph_\alpha, \Lambda_\kappa, \varphi) \geq \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{3\theta^{2\alpha}}) * \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{3\theta^{2\alpha+1}}) * \cdots * \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{3^{\kappa-1}\theta^{2\kappa-2}}$$
$$* \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{3^{\kappa-1}\theta^{2\kappa-1}}) * \Gamma(\aleph_0, \Lambda_0, \frac{\varphi}{3^{\kappa-1}\theta^{2\kappa}})$$

and

$$\Upsilon(\aleph_\alpha, \Lambda_\kappa, \varphi) \leq \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{3\theta^{2\alpha}}) \circ \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{3\theta^{2\alpha+1}}) \circ \cdots \circ \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{3^{\kappa-1}\theta^{2\kappa-2}}$$
$$\circ \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{3^{\kappa-1}\theta^{2\kappa-1}}) \circ \Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{3^{\kappa-1}\theta^{2\kappa}}).$$

As $\alpha, \kappa \to +\infty$, we deduce

$$\lim_{\alpha,\kappa\to+\infty} \Gamma(\aleph_\alpha, \Lambda_\kappa, \varphi) = 1 * 1 * \cdots * 1 = 1$$

and

$$\lim_{\alpha,\kappa\to+\infty} \Upsilon(\aleph_\alpha, \Lambda_\kappa, \varphi) = 0 \circ 0 \circ \cdots \circ 0 = 0.$$

Which implies that bisequence $(\aleph_\alpha, \Lambda_\alpha)$ is a Cauchy bisequence. Since $(\Omega, \Theta, \Gamma, \Upsilon, *, \circ)$ is a complete intuitionistic fuzzy bipolar metric space. Then, $\{\aleph_\alpha\} \to \aleph$ and $\{\Lambda_\alpha\} \to \aleph$, where $\aleph \in \Omega \cap \Theta$. Using 3.1 and 3.1, we get

$$\Gamma(\aleph, \mathfrak{p}\aleph, \varphi) \geq \Gamma(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}) * \Gamma(\aleph_{\alpha+1}, \aleph_{\alpha+1}, \frac{\varphi}{3}) * \Gamma(\aleph_{\alpha+1}, \mathfrak{p}\aleph, \frac{\varphi}{3})$$

$$= \Gamma(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}) * \Gamma(\mathfrak{p}\aleph_\alpha, \mathfrak{p}\aleph_\alpha, \frac{\varphi}{3}) * \Gamma(\mathfrak{p}\aleph_\alpha, \mathfrak{p}\aleph, \frac{\varphi}{3})$$

$$\to 1 * 1 * 1 = 1 \quad \text{as} \quad \alpha \to +\infty,$$

and

$$\Upsilon(\aleph, \mathfrak{p}\aleph, \varphi) \leq \Upsilon(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \mathfrak{p}\aleph, \frac{\varphi}{3}$$
$$= \Upsilon(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \Upsilon(\mathfrak{p}\aleph_{\alpha+1}, \mathfrak{p}\aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \Upsilon(\mathfrak{p}\aleph_{\alpha+1}, \mathfrak{p}\aleph, \frac{\varphi}{3})$$

$$\to 0 \circ 0 \circ 0 = 0 \quad \text{as} \quad \alpha \to +\infty.$$

Hence, $\mathfrak{p}\aleph = \aleph$.
Let $\mathfrak{p}\omega = \omega$ for some $\omega \in \Omega \cap \Theta$, then

$$1 \geq \Gamma(\omega, \aleph, \varphi) = \Gamma(\mathfrak{p}\aleph, \mathfrak{p}\omega, \varphi) \geq \Gamma(\omega, \aleph, \frac{\varphi}{\theta}) = \Gamma(\mathfrak{p}\aleph, \mathfrak{p}\omega, \frac{\varphi}{\theta}) \geq \Gamma(\omega, \aleph, \frac{\varphi}{\theta^2}) \geq \cdots$$
$$\geq \Gamma(\omega, \aleph, \frac{\varphi}{\theta^\alpha}) \to 1 \quad \text{as} \quad \alpha \to +\infty,$$

$$0 \leq \Upsilon(\omega, \aleph, \varphi) = \Upsilon(\mathfrak{p}\aleph, \mathfrak{p}\omega, \varphi) \leq \Upsilon(\omega, \aleph, \frac{\varphi}{\theta}) = \Upsilon(\mathfrak{p}\aleph, \mathfrak{p}\omega, \frac{\varphi}{\theta}) \leq \Upsilon(\omega, \aleph, \frac{\varphi}{\theta^2}) \leq \cdots$$
$$\leq \Upsilon(\omega, \aleph, \frac{\varphi}{\theta^\alpha}) \to 0 \quad \text{as} \quad \alpha \to +\infty,$$

by using 3.1 and 3.1, we get $\aleph = \omega$.
Definition 3.4 Let $(\Omega, \Theta, \Gamma, \Upsilon, *, \circ)$ be a intuitionistic fuzzy bipolar metric space. A map $\mathfrak{p}: \Omega \cup \Theta \to \Omega \cup \Theta$ is an IFB(intuitionistic fuzzy bipolar)-contraction if we can find $0 < \theta < 1$ satisfying

$$\frac{1}{\Gamma(\mathfrak{p}\aleph, \mathfrak{p}\Lambda, \varphi)} - 1 \leq \theta[\frac{1}{\Gamma(\aleph, \Lambda, \varphi)} - 1] \tag{7}$$

and

$$\Upsilon(\mathfrak{p}\aleph, \mathfrak{p}\Lambda, \varphi) \leq \theta\Upsilon(\aleph, \Lambda, \varphi), \tag{8}$$

for all $\aleph \in \Omega$, $\Lambda \in \Theta$ and $\varphi > 0$.

Theorem 3.3 *Let* $(\Omega, \Theta, \Gamma, \Upsilon, *, \circ)$ *be a complete intuitionistic fuzzy bipolar metric space. Let* $\mathfrak{p}: \Omega \cup \Theta \to \Omega \cup \Theta$ *be a mapping satisfyig:*

1. $\mathfrak{p}(\Omega) \subseteq \Omega$ and $\mathfrak{p}(\Theta) \subseteq \Theta$;

2. $\mathfrak{p}$ is IFB-contraction, for all $\aleph \in \Omega$, $\Lambda \in \Theta$ and $\varphi > 0$.

Then, $\mathfrak{p}$ has a unique fixed point.
Proof. Let $\aleph_0 \in \Omega$ and $\Lambda_0 \in \Theta$ and assume that $\mathfrak{p}(\aleph_\alpha) = \aleph_{\alpha+1}$ and $\mathfrak{p}(\Lambda_\alpha) = \Lambda_{\alpha+1}$ for all $\alpha \in \mathbb{N} \cup \{0\}$. Then we get $(\aleph_\alpha, \Theta_\alpha)$ as a bisequence on intuitionistic fuzzy bipolar metric space $(\Omega, \Theta, \Gamma, \Upsilon, *, \circ)$. By using (7) and (8) for all $\varphi > 0$, we deduce

$$\frac{1}{\Gamma(\aleph_\alpha, \Lambda_\alpha, \varphi)} - 1 = \frac{1}{\Gamma(\mathfrak{p}\aleph_{\alpha-1}, \mathfrak{p}\Lambda_{\alpha-1}, \varphi)} - 1 \leq \theta[\frac{1}{\Gamma(\aleph_{\alpha-1}, \Lambda_{\alpha-1}, \varphi)}]$$
$$= \frac{\theta}{\Gamma(\aleph_{\alpha-1}, \Lambda_{\alpha-1}, \varphi)} - \theta$$

$$\Rightarrow \frac{1}{\Gamma(\aleph_\alpha, \Lambda_\alpha, \varphi)} \leq \frac{\theta}{\Gamma(\aleph_{\alpha-1}, \Lambda_{\alpha-1}, \varphi)} + (1-\theta) \leq \frac{\theta^2}{\Gamma(\aleph_{\alpha-2}, \Lambda_{\alpha-2}, \varphi)} + \theta(1-\theta) + (1-\theta)$$

Continuing this way, we deduce

$$\frac{1}{\Gamma(\aleph_\alpha, \Lambda_\alpha, \varphi)} \leq \frac{\theta^\alpha}{\Gamma(\aleph_0, \Lambda_0, \varphi)} + \theta^{\alpha-1}(1-\theta) + \theta^{\alpha-2}(1-\theta) + \cdots + \theta(1-\theta) + (1-\theta)$$
$$\leq \frac{\theta^\alpha}{\Gamma(\aleph_0, \Lambda_0, \varphi)} + (\theta^{\alpha-1} + \theta^{\alpha-2} + \cdots + 1)(1-\theta)$$
$$\leq \frac{\theta^\alpha}{\Gamma(\aleph_0, \Lambda_0, \varphi)} + (1-\theta^\alpha).$$

We obtain

$$\frac{1}{\frac{\theta^\alpha}{\Gamma(\aleph_0, \Lambda_0, \varphi)} + (1-\theta^\alpha)} \leq \Gamma(\aleph_\alpha, \Lambda_\alpha, \varphi) \tag{9}$$

$$\Upsilon(\aleph_\alpha, \Lambda_\alpha, \varphi) = \Upsilon(\mathfrak{p}\aleph_{\alpha-1}, \mathfrak{p}\Lambda_{\alpha-1}, \varphi) \leq \theta\Upsilon(\aleph_{\alpha-1}, \Lambda_{\alpha-1}, \varphi) = \Upsilon(\mathfrak{p}\aleph_{\alpha-2}, \mathfrak{p}\Lambda_{\alpha-2}, \varphi)$$
$$\leq \theta^2\Upsilon(\aleph_{\alpha-2}, \aleph_{\Lambda-2}, \varphi) \leq \cdots \leq \theta^\alpha\Upsilon(\aleph_0, \Lambda_0, \varphi), \tag{10}$$

and

$$\frac{1}{\frac{\theta^\alpha}{\Gamma(\aleph_1, \Lambda_0, \varphi)} + (1-\theta^\alpha)} \leq \Gamma(\aleph_{\alpha+1}, \Lambda_\alpha, \varphi) \tag{11}$$

$$\Upsilon(\aleph_{\alpha+1}, \Lambda_\alpha, \varphi) = \Upsilon(\mathfrak{p}\aleph_\alpha, \mathfrak{p}\Lambda_{\alpha-1}, \varphi) \leq \theta\Upsilon(\aleph_\alpha, \Lambda_{\alpha-1}, \varphi) = \Upsilon(\mathfrak{p}\aleph_{\alpha-1}, \mathfrak{p}\Lambda_{\alpha-2}, \varphi)$$
$$\leq \theta^2\Upsilon(\aleph_{\alpha-1}, \aleph_{\Lambda-2}, \varphi) \leq \cdots \leq \theta^\alpha\Upsilon(\aleph_1, \Lambda_0, \varphi). \tag{12}$$

Let $\alpha < \kappa$, for $\alpha, \kappa \in \mathbb{N}$. Then,

$$\Gamma(\aleph_\alpha, \Lambda_\kappa, \varphi) \geq \Gamma(\aleph_\alpha, \Lambda_\alpha, \frac{\varphi}{3}) * \Gamma(\aleph_{\alpha+1}, \Lambda_\alpha, \frac{\varphi}{3}) * \Gamma(\aleph_{\alpha+1}, \Lambda_\kappa, \frac{\varphi}{3})$$
$$\vdots$$
$$\geq \Gamma(\aleph_\alpha, \Lambda_\alpha, \frac{\varphi}{3}) * \Gamma(\aleph_{\alpha+1}, \Lambda_\alpha, \frac{\varphi}{3}) * \cdots * \Gamma(\aleph_{\kappa-1}, \Lambda_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}) * \Gamma(\aleph_\kappa, \Lambda_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}$$
$$* \Gamma(\aleph_\kappa, \Lambda_\kappa, \frac{\varphi}{3^{\kappa-1}}),$$

and

$$\Upsilon(\aleph_\alpha, \Lambda_\kappa, \varphi) \leq \Upsilon(\aleph_\alpha, \Lambda_\alpha, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \Lambda_\alpha, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \Lambda_\kappa, \frac{\varphi}{3})$$
$$\vdots$$
$$\leq \Upsilon(\aleph_\alpha, \Lambda_\alpha, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \Lambda_\alpha, \frac{\varphi}{3}) \circ \cdots \circ \Upsilon(\aleph_{\kappa-1}, \Lambda_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}) \circ \Upsilon(\aleph_\kappa, \Lambda_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}})$$
$$\circ \Upsilon(\aleph_\kappa, \Lambda_\kappa, \frac{\varphi}{3^{\kappa-1}}).$$

Therefore,

$$\begin{aligned}\varGamma(\aleph_\alpha,\varLambda_\kappa,\varphi) \geq\ & \varGamma(\aleph_\alpha,\varLambda_\alpha,\frac{\varphi}{3}) * \varGamma(\aleph_{\alpha+1},\varLambda_\alpha,\frac{\varphi}{3}) * \cdots * \varGamma(\aleph_{\kappa-1},\varLambda_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}) \\ & * \varGamma(\aleph_\kappa,\varLambda_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}) * \varGamma(\aleph_\kappa,\varLambda_\kappa,\frac{\varphi}{3^{\kappa-1}}) \\ & \geq \frac{1}{\frac{\theta^\alpha}{\varGamma(\aleph_0,\varLambda_0,\frac{\varphi}{3})}+(1-\theta^\alpha)} * \frac{1}{\frac{\theta^\alpha}{\varGamma(\aleph_1,\varLambda_0,\frac{\varphi}{3})}+(1-\theta^\alpha)} * \cdots \\ & * \frac{1}{\frac{\theta^{\kappa-1}}{\varGamma(\aleph_0,\varLambda_0,\frac{\varphi}{3^{\kappa-1}})}+(1-\theta^{\kappa-1})} * \frac{1}{\frac{\theta^{\kappa-1}}{\varGamma(\aleph_1,\varLambda_0,\frac{\varphi}{3^{\kappa-1}})}+(1-\theta^{\kappa-1})} \\ & * \frac{1}{\frac{\theta^{\kappa}}{\varGamma(\aleph_0,\varLambda_0,\frac{\varphi}{3^{\kappa-1}})}+(1-\theta^{\kappa})},\end{aligned}$$

And

$$\begin{aligned}\varUpsilon(\aleph_\alpha,\varLambda_\kappa,\varphi) \leq\ & \varUpsilon(\aleph_\alpha,\varLambda_\alpha,\frac{\varphi}{3}) \circ \varUpsilon(\aleph_{\alpha+1},\varLambda_\alpha,\frac{\varphi}{3}) \circ \cdots \circ \varUpsilon(\aleph_{\kappa-1},\varLambda_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}) \\ & \circ \varUpsilon(\aleph_\kappa,\varLambda_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}) \circ \varUpsilon(\aleph_\kappa,\varLambda_\kappa,\frac{\varphi}{3^{\kappa-1}}) \\ & \leq \theta^\alpha\varUpsilon(\aleph_0,\varLambda_0,\frac{\varphi}{3}) \circ \theta^\alpha\varUpsilon(\aleph_1,\varLambda_0,\frac{\varphi}{3^{\kappa-1}}) \circ \cdots \circ \theta^{\kappa-1}\varUpsilon(\aleph_0,\varLambda_0,\frac{\varphi}{3^{\kappa-1}}) \\ & \circ \theta^{\kappa-1}\varUpsilon(\aleph_1,\varLambda_0,\frac{\varphi}{3^{\kappa-1}}) \circ \theta^{\kappa}\varUpsilon(\aleph_0,\varLambda_0,\frac{\varphi}{3^{\kappa-1}}).\end{aligned}$$

Which implies that,

$$\begin{aligned}\varGamma(\aleph_\alpha,\varLambda_\kappa,\varphi) \geq\ & \frac{1}{\frac{\theta^\alpha}{\varGamma(\aleph_0,\varLambda_0,\frac{\varphi}{3})}+(1-\theta^\alpha)} * \frac{1}{\frac{\theta^\alpha}{\varGamma(\aleph_1,\varLambda_0,\frac{\varphi}{3})}+(1-\theta^\alpha)} * \cdots \\ & * \frac{1}{\frac{\theta^{\kappa-1}}{\varGamma(\aleph_0,\varLambda_0,\frac{\varphi}{3^{\kappa-1}})}+(1-\theta^{\kappa-1})} * \frac{1}{\frac{\theta^{\kappa-1}}{\varGamma(\aleph_1,\varLambda_0,\frac{\varphi}{3^{\kappa-1}})}+(1-\theta^{\kappa-1})} \\ & * \frac{1}{\frac{\theta^{\kappa}}{\varGamma(\aleph_0,\varLambda_0,\frac{\varphi}{3^{\kappa-1}})}+(1-\theta^{\kappa})},\end{aligned}$$

and

$$\begin{aligned}\varUpsilon(\aleph_\alpha,\varLambda_\kappa,\varphi) \leq\ & \theta^\alpha\varUpsilon(\aleph_0,\varLambda_0,\frac{\varphi}{3}) \circ \theta^\alpha\varUpsilon(\aleph_1,\varLambda_0,\frac{\varphi}{3^{\kappa-1}}) \circ \cdots \circ \theta^{\kappa-1}\varUpsilon(\aleph_0,\varLambda_0,\frac{\varphi}{3^{\kappa-1}} \\ & \circ \theta^{\kappa-1}\varUpsilon(\aleph_1,\varLambda_0,\frac{\varphi}{3^{\kappa-1}}) \circ \theta^{\kappa}\varUpsilon(\aleph_0,\varLambda_0,\frac{\varphi}{3^{\kappa-1}}).\end{aligned}$$

As $\alpha,\kappa \to +\infty$, we deduce

$$\lim_{\alpha,\kappa\to+\infty} \varGamma(\aleph_\alpha,\varLambda_\kappa,\varphi) = 1 * 1 * \cdots * 1 = 1,$$

and

$$\lim_{\alpha,\kappa\to+\infty} \varUpsilon(\aleph_\alpha,\varLambda_\kappa,\varphi) = 0 \circ 0 \circ \cdots \circ 0 = 0.$$

Which implies that bisequence $(\aleph_\alpha, \Lambda_\alpha)$ is a Cauchy bisequence. Since $(\Omega, \Theta, \Gamma, \Upsilon, *, \circ)$ is a complete intuitionistic fuzzy bipolar metric space. Then, $\{\aleph_\alpha\} \to \aleph$ and $\{\Lambda_\alpha\} \to \aleph$, where $\aleph \in \Omega \cap \Theta$. Using 3.1 and 3.1, we get

$$\begin{aligned}\Gamma(\aleph, \mathfrak{p}\aleph, \varphi) &\geq \Gamma\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) * \Gamma\left(\aleph_{\alpha+1}, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) * \Gamma\left(\aleph_{\alpha+1}, \mathfrak{p}\aleph, \frac{\varphi}{3}\right) \\ &= \Gamma\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) * \Gamma\left(\mathfrak{p}\aleph_\alpha, \mathfrak{p}\aleph_\alpha, \frac{\varphi}{3}\right) * \Gamma\left(\mathfrak{p}\aleph_\alpha, \mathfrak{p}\aleph, \frac{\varphi}{3}\right) \\ &\geq \Gamma\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) * \frac{1}{\frac{\theta^{\alpha+1}}{\Gamma\left(\aleph_0, \Lambda_0, \frac{\varphi}{3}\right)} + (1-\theta^{\alpha+1})} * \Gamma\left(\mathfrak{p}\aleph_\alpha, \mathfrak{p}\aleph, \frac{\varphi}{3}\right)\end{aligned}$$

$$\to 1 * 1 * 1 = 1 \quad \text{as} \quad \alpha \to +\infty,$$

and

$$\begin{aligned}\Upsilon(\aleph, \mathfrak{p}\aleph, \varphi) &\leq \Upsilon(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \Upsilon(\aleph_{\alpha+1}, \mathfrak{p}\aleph, \frac{\varphi}{3}) \\ &= \Upsilon(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \Upsilon(\mathfrak{p}\aleph_{\alpha+1}, \mathfrak{p}\aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \Upsilon(\mathfrak{p}\aleph_{\alpha+1}, \mathfrak{p}\aleph, \frac{\varphi}{3}) \\ &\leq \Upsilon(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}) \circ \theta^{\alpha+1}\Upsilon(\aleph_0, \Lambda_0, \frac{\varphi}{3}) \circ \Upsilon(\mathfrak{p}\aleph_{\alpha+1}, \mathfrak{p}\aleph, \frac{\varphi}{3})\end{aligned}$$

$$\to 0 \circ 0 \circ 0 = 0 \quad \text{as} \quad \alpha \to +\infty.$$

Hence, $\mathfrak{p}\aleph = \aleph$. Let $\mathfrak{p}\omega = \omega$ for some $\omega \in \Omega$, then

$$\frac{1}{\Gamma(\aleph, \omega, \varphi)} - 1 = \frac{1}{\Gamma(\mathfrak{p}\aleph, \mathfrak{p}\omega, \varphi)} - 1 \leq \theta[\frac{1}{\Gamma(\aleph, \omega, \varphi)} - 1] < \frac{1}{\Gamma(\aleph, \omega, \varphi)} - 1,$$

which is a contradiction.

$$\Upsilon(\aleph, \omega, \varphi) = \Upsilon(\mathfrak{p}\aleph, \mathfrak{p}\omega, \varphi) \leq \theta\Upsilon(\aleph, \omega, \varphi) < \Upsilon(\aleph, \omega, \varphi),$$

which is a contradiction as well. Therefore, $\Gamma(\aleph, \omega, \varphi) = 1, \Upsilon(\aleph, \omega, \varphi) = 0$, hence, $\aleph = \omega$.

Example 3.2 Let $\Omega = [0,1]$ *and* $\Theta = \{0\} \cup \mathbb{N} - \{1\}$*. Define* $\Gamma, \Upsilon, : \Omega \times \Theta \times (0, +\infty) \to [0,1]$ *as*

$$\Gamma(\aleph, \Lambda, \varphi) = \frac{\varphi}{\varphi + |\aleph - \Lambda|}, \Upsilon(\aleph, \Lambda, \varphi) = \frac{|\aleph - \Lambda|}{\varphi + |\aleph - \Lambda|}.$$

Then, $(\Omega, \Theta, \Gamma, \Upsilon, *, \circ)$ is a complete intuitionistic fuzzy bipolar metric space with CTN $\nu * \xi = \nu\xi$ and CTCN $\nu \circ \xi = \max\{\nu, \xi\}$.

Define $\mathfrak{p}: (\Omega, \Theta, \Gamma, \Upsilon, *, \circ) \rightrightarrows (\Omega, \Theta, \Gamma, \Upsilon, *, \circ)$ by

$$\mathfrak{p}(\aleph) = \begin{cases} \frac{1 - 3^{-\aleph}}{5}, & \text{if} \aleph \in [0,1], \\ 0, & \text{if} \aleph \in \mathbb{N} - \{1\}, \end{cases}$$

for all $\aleph \in \Omega \cup \Theta$ and take $\theta \in [\frac{1}{2}, 1)$, then

$$\Gamma(\mathfrak{p}\aleph,\mathfrak{p}\Lambda,\theta\varphi)=\Gamma(\frac{1-3^{-\aleph}}{5},\frac{1-3^{-\Lambda}}{5},\theta\varphi)=\frac{\theta\varphi}{\theta\varphi+|\frac{1-3^{-\aleph}}{5}-\frac{1-3^{-\Lambda}}{5}|}$$
$$=\frac{\theta\varphi}{\theta\varphi+\frac{|3^{-\aleph}-3^{-\Lambda}|}{5}}\geq\frac{\theta\varphi}{\theta\varphi+\frac{|\aleph-\Lambda|}{5}}=\frac{5\theta\varphi}{5\theta\varphi+|\aleph-\Lambda|}\geq\frac{\varphi}{\varphi+|\aleph-\Lambda|}$$
$$=\Gamma(\aleph,\Lambda,\varphi),$$
$$\Upsilon(\mathfrak{p}\aleph,\mathfrak{p}\Lambda,\theta\varphi)=\Upsilon(\frac{1-3^{-\aleph}}{5},\frac{1-3^{-\Lambda}}{5},\theta\varphi)=\frac{|\frac{1-3^{-\aleph}}{5}-\frac{1-3^{-\Lambda}}{5}|}{\theta\varphi+|\frac{1-3^{-\aleph}}{5}-\frac{1-3^{-\Lambda}}{5}|}$$
$$=\frac{\frac{|3^{-\aleph}-3^{-\Lambda}|}{5}}{\theta\varphi+\frac{|3^{-\aleph}-3^{-\Lambda}|}{5}}=\frac{|3^{-\aleph}-3^{-\Lambda}|}{5\theta\varphi+|3^{-\aleph}-3^{-\Lambda}|}\leq\frac{|\aleph-\Lambda|}{5\theta\varphi+|\aleph-\Lambda|}$$
$$\leq\frac{|\aleph-\Lambda|}{\varphi+|\aleph-\Lambda|}=\Upsilon(\aleph,\Lambda,\varphi).$$

Therefore, the conditions of Theorem 3.1 are fulfilled, and 0 is the only fixed point for $\mathfrak{p}$.

4. Application

Let $\Omega=\mathcal{C}([\mathfrak{c},\mathfrak{a}],[0,+\infty))$ is the set of all continuous functions defined on $[\mathfrak{c},\mathfrak{a}]$ with values in the interval $[0,+\infty)$ and $\Theta=\mathcal{C}([\mathfrak{c},\mathfrak{a}],(-\infty,0])$ is the set of all continuous functions defined on $[\mathfrak{c},\mathfrak{a}]$ with values in the interval $(-\infty,0]$.
Suppose the integral equation:

$$\aleph(\chi)=\wedge(\chi)+\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\Xi(\chi,\rho)\aleph(\chi)d\rho \quad \text{for} \quad \chi,\rho\in[\mathfrak{c},\mathfrak{a}]d \tag{13}$$

where $\delta>0,\wedge(\rho)$ is a fuzzy function of $\rho:\rho\in[\mathfrak{c},\mathfrak{a}]$ and $\Xi:\mathcal{C}([\mathfrak{c},\mathfrak{a}]\times\mathbb{R})\to\mathbb{R}^{+}$. Define Γ and Υ by

$$\Gamma(\aleph(\chi),\Lambda(\chi),\varphi)=\sup_{\chi\in[\mathfrak{c},\mathfrak{a}]}\frac{\varphi}{\varphi+|\aleph(\chi)-\Lambda(\chi)|} \quad \text{forall} \quad \aleph,\Lambda\in\Omega \text{ and } \varphi>0,$$

and

$$\Upsilon(\aleph(\chi),\Lambda(\chi),\varphi)=1-\sup_{\chi\in[\mathfrak{c},\mathfrak{a}]}\frac{\varphi}{\varphi+|\aleph(\chi)-\Lambda(\chi)|} \quad \text{forall} \quad \aleph,\Lambda\in\Omega \text{ and } \varphi>0,$$

with CTN and CTCN define by $\nu*\xi=\nu\cdot\xi$ and $\nu\circ\xi=\max\{\nu,\xi\}$. Then $(\Omega,\Theta,\Gamma,\Upsilon,*,\circ)$ is a complete intuitionistic fuzzy bipolar metric space.
Suppose that $|\Xi(\chi,\rho)\aleph(\chi)-\Xi(\chi,\rho)\Lambda(\chi)|\leq|\aleph(\chi)-\Lambda(\chi)|$ for $\aleph\in\Omega,\Lambda\in\Theta,\theta\in(0,1)$ and $\forall\chi,\rho\in[\mathfrak{c},\mathfrak{a}]$. Also, let $\Xi(\chi,\rho)(\delta\int_{\mathfrak{c}}^{\mathfrak{a}}d\rho)\leq\theta<1$. Then, the integral Equation (13) has a unique solution.
Proof. Define $\mathfrak{p}:(\Omega,\Theta,\Gamma,\Upsilon,*,\circ)\rightrightarrows(\Omega,\Theta,\Gamma,\Upsilon,*,\circ)$ by

$$\mathfrak{p}\aleph(\chi)=\wedge(\chi)+\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\Xi(\chi,\rho)\aleph(\chi)d\rho \quad \text{forall} \quad \chi,\rho\in[\mathfrak{c},\mathfrak{a}].$$

Now, for all $\aleph,\Lambda\in\Omega\cup\Theta$, we deduce

$$\Gamma(\mathfrak{p}\aleph(\chi),\mathfrak{p}\Lambda(\chi),\theta\varphi)=\sup_{\chi\in[\mathrm{c},\mathrm{a}]}\frac{\theta\varphi}{\theta\varphi+|\mathfrak{p}\aleph(\chi)-\mathfrak{p}\Lambda(\chi)|}$$
$$=\sup_{\chi\in[\mathrm{c},\mathrm{a}]}\frac{\theta\varphi}{\theta\varphi+|\wedge(\chi)+\delta\int_{\mathrm{c}}^{\mathrm{a}}\Xi(\chi,\rho)\aleph(\chi)\mathrm{d}\rho-\wedge(\chi)-\delta\int_{\mathrm{c}}^{\mathrm{a}}\Xi(\chi,\rho)\aleph(\chi)\mathrm{d}\rho|}$$
$$=\sup_{\chi\in[\mathrm{c},\mathrm{a}]}\frac{\theta\varphi}{\theta\varphi+|\delta\int_{\mathrm{c}}^{\mathrm{a}}\Xi(\chi,\rho)\aleph(\chi)\mathrm{d}\rho-\delta\int_{\mathrm{c}}^{\mathrm{a}}\Xi(\chi,\rho)\aleph(\chi)\mathrm{d}\rho|}$$
$$=\sup_{\chi\in[\mathrm{c},\mathrm{a}]}\frac{\theta\varphi}{\theta\varphi+|\Xi(\chi,\rho)\aleph(\chi)-\Xi(\chi,\rho)\Lambda(\chi)|(\delta\int_{\mathrm{c}}^{\mathrm{a}}\mathrm{d}\rho)}\geq\sup_{\chi\in[\mathrm{c},\mathrm{a}]}\frac{\varphi}{\varphi+|\aleph(\chi)-\Lambda(\chi)|}$$
$$\geq\Gamma(\aleph(\chi),\Lambda(\chi),\varphi),$$

and

$$\Upsilon(\mathfrak{p}\aleph(\chi),\mathfrak{p}\Lambda(\chi),\theta\varphi)=1-\sup_{\chi\in[\mathrm{c},\mathrm{a}]}\frac{\theta\varphi}{\theta\varphi+|\mathfrak{p}\aleph(\chi)-\mathfrak{p}\Lambda(\chi)|}$$
$$=1-\sup_{\chi\in[\mathrm{c},\mathrm{a}]}\frac{\theta\varphi}{\theta\varphi+|\wedge(\chi)+\delta\int_{\mathrm{c}}^{\mathrm{a}}\Xi(\chi,\rho)\aleph(\chi)\mathrm{d}\rho-\wedge(\chi)-\delta\int_{\mathrm{c}}^{\mathrm{a}}\Xi(\chi,\rho)\aleph(\chi)\mathrm{d}\rho|}$$
$$=1-\sup_{\chi\in[\mathrm{c},\mathrm{a}]}\frac{\theta\varphi}{\theta\varphi+|\delta\int_{\mathrm{c}}^{\mathrm{a}}\Xi(\chi,\rho)\aleph(\chi)\mathrm{d}\rho-\delta\int_{\mathrm{c}}^{\mathrm{a}}\Xi(\chi,\rho)\aleph(\chi)\mathrm{d}\rho|}$$
$$=1-\sup_{\chi\in[\mathrm{c},\mathrm{a}]}\frac{\theta\varphi}{\theta\varphi+|\Xi(\chi,\rho)\aleph(\chi)-\Xi(\chi,\rho)\Lambda(\chi)|(\delta\int_{\mathrm{c}}^{\mathrm{a}}\mathrm{d}\rho)}\leq1-\sup_{\chi\in[\mathrm{c},\mathrm{a}]}\frac{\varphi}{\varphi+|\aleph(\chi)-\Lambda(\chi)|}$$
$$\leq\Upsilon(\aleph(\chi),\Lambda(\chi),\varphi).$$

Therefore, the conditions of Theorem 3.1 are fulfilled and operator $\mathfrak{p}$ has a unique fixed point.

Example 4.1 *Assume the following integral equation.*

$$\aleph(\chi)=|\sin\chi|+\frac{1}{9}\int_0^1\rho\aleph(\rho)\mathrm{d}\rho,\quad \text{forall}\quad \rho\in[0,1]$$

Then it has a unique solution in $\Omega\cup\Theta$.

Proof. Let $\mathfrak{p}:(\Omega,\Theta,\Gamma,\Upsilon,*,\circ)\rightrightarrows(\Omega,\Theta,\Gamma,\Upsilon,*,\circ)$ be defined by

$$\mathfrak{p}\aleph(\chi)=|\sin\chi|+\frac{1}{9}\int_0^1\rho\aleph(\rho)\mathrm{d}\rho,$$

and set $\Xi(\chi,\rho)\aleph(\chi)=\frac{1}{9}\rho\aleph(\rho)$ and $\Xi(\chi,\rho)\Lambda(\chi)=\frac{1}{9}\rho\Lambda(\rho)$, where $\aleph,\Lambda\in\Omega\cup\Theta$, and for all $\chi,\rho\in[0,1]$. Then we have

$$|\Xi(\chi,\rho)\aleph(\chi)-\Xi(\chi,\rho)\Lambda(\chi)|=|\frac{1}{9}\rho\aleph(\rho)-\frac{1}{9}\rho\Lambda(\rho)|=\frac{\rho}{9}|\aleph(\rho)-\Lambda(\rho)|$$
$$\leq|\aleph(\rho)-\Lambda(\rho)|,$$

And $\frac{1}{9}\int_0^1\rho\mathrm{d}\rho=\frac{1}{9}(\frac{(1)^2}{2}-\frac{(0)^2}{2})=\frac{1}{9}=\theta<1$, where $\delta=\frac{1}{9}$. Therefore, the conditions of Theorem 3.1 are fulfilled. Hence $\mathfrak{p}$ has a unique solution in $\Omega\cup\Theta$.

5. Conclusion

In this paper, we introduced the concept of intuitionistic fuzzy bipolar metric space and proved fixed point theorems. Readers can explore extending the results in the setting of Chatterjea, Hardy Rogers, Ciric' and Suzuki contraction types. The authors received no specific funding for this study. The authors declare that they have no conflicts of interest to report regarding the present study.

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